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EXISTENCE OF SOLUTIONS FOR THE SYSTEM OF EQUATIONS DESCRIBING THE FILTRATION OF A BURNING GAS

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A two-temperature model describing the propagation of combustion waves in a chemically inert porous medium was discussed in [1] for the filtration of a combustible gas mixture. The approximate solution obtained there described satisfactorily the experimental results obtained at the Institute of Chemical Kinetics and Combustion (Siberian Branch, Academy of Sciences of the USSR). The physical basis of the process is the recovery of energy from conductive transport in the solid structure and heat exchange between the phases. Pressure gradients are ignored in the treatment; this corresponds to experimental conditions.

One of the questions which arises in a qualitative study of the model is the existence of solutions of the corresponding system of equations. There are several papers in which the propagation problem for an exothermal reaction front is solved for a one-temperature model (cf. [2] and the bibliography given there).

For particular assumptions on the analogy between the concentration and temperature fields and the reaction rate function (see [3]), where the monotonicity of the solution as a function of the wave velocity was used in an essential way. In the present paper, the existence of the solution to the equations governing the process referred to above will be demonstrated. It turns out that the solution is not monotonic with respect to the wave velocity, and thus the proof of [3] does not apply. We give an asymptotic formula for the wave velocity which corresponds to the approximate solution of [1].

1. Statement of the Problem. As in [1], the steady-state equations for the propagation of combustion waves in an inert, porous medium for the filtration of a combustible gas mixture have the form

$$\begin{gathered} a_\Theta\, d^2\Theta/dx^2 + u d\Theta/dx + \alpha_\Theta(T-\Theta) = 0, \\ (v-u)dT/dx + \alpha_T(T-\Theta) = (Q/c_T)w(n,\,T), \\ (v-u)dn/dx = -w(n,\,T),\ u \neq v, \end{gathered} \tag{1.1}$$

where Θ and T are the temperatures of the solid structure and gas, respectively; n, relative mass concentration of the solute; v, flow velocity; u, wave velocity; $\alpha_\Theta = \alpha_0 S/(1-\varepsilon)c_\Theta\rho_\Theta$; $\alpha_T = \sigma\alpha_\Theta$; $\sigma = (1-\varepsilon)c_\Theta\rho_\Theta/(\varepsilon c_T\rho_T)$; α_0, heat-exchange coefficient; S, specific area; ε, porosity constant; c_Θ, ρ_Θ and c_T, ρ_T, specific heats and densities of the solid structure and gas,

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respectively; a_Θ thermal conductivity of the solid; Q, heat of the reaction; and w(n, T), reaction rate function. The parameters v and ρ_T are taken from the initial mixture (from the equation of continuity). It is assumed that $\sigma \gg 1$, so that the thermal inertia of the solid phase is much greater than that of the gas. Hence we do not consider conductive transport in the gas in our model.

Equations (1.1) are considered for $-\infty < x < \infty$ with the conditions

$$x = -\infty,\ \Theta = T_0,\ T = T_0,\ n = 1; \tag{1.2a}$$

$$x = +\infty,\ \theta \text{ is the temperature of the solid structure},\ n = n_e < 1. \tag{1.2b}$$

The correctness of condition (1.2a) follows from the relation $w(n, T_0) = 0$. Only for this case is the solution at $x = -\infty$ a stationary point of (1.1). We will assume the following conditions for the reaction rate function

$$\begin{gathered} w(n, T) = 0,\ T \leqslant T_1,\ T_1 > T_0, \\ w(n, T) > 0,\ T > T_1,\ n > 0,\ w(0, T) = 0. \end{gathered} \tag{1.3}$$

The above conditions imply that there exists a straight line segment $\Theta = T$ of stationary points on which the point $\Theta = T_0$, $T = T_0$ is included. From (1.2a), we can lower the order of the system (1.1). Multiplying the first equation by $-\sigma$, the third by Q/c_T, adding all three equations and integrating the result from $-\infty$ to x, we find the following system of equations

$$\begin{gathered} \frac{d\Theta}{dx} = -\frac{u}{a_\Theta}(\Theta - T_0) + \frac{v-u}{\sigma a_\Theta}\left(T + \frac{Q}{c_T}n - T_0 - \frac{Q}{c_T}\right), \\ \frac{dT}{dx} = \frac{\alpha_T}{v-u}(\Theta - T) - \frac{Q}{c_T}\frac{dn}{dx},\ \frac{dn}{dx} = -\frac{1}{v-u}w(n, T). \end{gathered} \tag{1.4}$$

We will assume that $T(0, u) = T_1$ in (1.4). Then the solution for $x < 0$ can be written in the form

$$\begin{gathered} \Theta(x, u) = T_0 + r_1(T_1 - T_0)e^{\mu_1 x}, \\ T(x, u) = T_0 + (T_1 - T_0)e^{\mu_1 x},\ n(x, u) = 1, \end{gathered}$$

where $\mu_1 > 0$. Here we use the notation $r_i = 1 + (v - u)\mu_i/\alpha_T$, where μ_i are the roots of the characteristic equation

$$\mu^2 + \left(\frac{u}{a_\Theta} + \frac{\alpha_T}{v-u}\right)\mu - \frac{\alpha_\Theta}{a_\Theta}\frac{v-(1+\sigma)u}{v-u} = 0.$$

In order that there exist a positive root μ_1 (the roots of the equation are real), the inequality $u < v/(1 + \sigma)$ must be satisfied. Note that the inequality $u > v$ is contrary to the result of integrating the third equation of (1.4) from $-\infty$ to $+\infty$ with the use of condition (1.3). Hence we are led to a Cauchy problem for (1.4) with the initial conditions

$$\Theta(0, u) = T_0 + r_1(T_1 - T_0),\ T(0, u) = T_1,\ n(0, u) = 1. \tag{1.5}$$

It is required to prove the existence of a $u < v/(1 + \sigma)$ such that the temperature Θ is bounded at $x \to +\infty$ [condition (1.2b)]. We will demonstrate that the functions Θ and T have finite bounds. The system (1.4) has an additional stationary point

$$\Theta = T = T_0 + \frac{Q}{c_T}\frac{v-u}{v-(1+\sigma)u},\quad n = 0. \tag{1.6}$$

It will be shown that the solution is bounded as this point is approached. The dependence of the stationary point on the wave velocity is a function of the details of the model, unlike the classical case.

2. Existence of the Solution. We introduce variables y_1 and y_2, satisfying

$$r_1 y_1 + r_2 y_2 = \Theta - T_0,\ y_1 + y_2 = T + Qn/c_T - T_0 - Q/c_T. \tag{2.1}$$

This transformation is nonsingular because the determinant $r_1 - r_2 = [(v - u)/\alpha_T](\mu_1 - \mu_2) > 0$. The equations for y_1 and y_2 take the form

$$\begin{gathered} \frac{dy_1}{dx} = \mu_1 y_1 + \frac{r_2}{r_1 - r_2}\frac{\alpha_T}{v-u}\frac{Q}{c_T}(1-n), \\ \frac{dy_2}{dx} = \mu_2 y_2 - \frac{r_1}{r_1 - r_2}\frac{\alpha_T}{v-u}\frac{Q}{c_T}(1-n). \end{gathered} \tag{2.2}$$

According to (1.5), the initial conditions are

$$y_1(0, u) = T_1 - T_0,\ y_2(0, u) = 0. \tag{2.3}$$

We introduce the function $\varphi(u) = -\frac{v-u}{\alpha_T}\frac{\mu_1}{r_2}(r_1 - r_2)$ and the dimensionless parameter $q = (Q/c_T)/(T_1 - T_0)$. The solution of (2.2) and (2.3) satisfies the integral relations

$$y_1 = \frac{T_1 - T_0}{\varphi} e^{\mu_1 x}\left(\varphi - q\mu_1 \int_0^x e^{-\mu_1 s}(1-n)\,ds\right), \tag{2.4}$$

$$y_2 = \frac{T_1 - T_0}{\varphi} q \frac{r_1}{r_2} \mu_1 \int_0^x e^{\mu_2(x-s)}(1-n)\,ds.$$

Because $0 \le n(x, u) \le 1$ [this follows from (1.3)] and $r_2 < 0$ (this follows from the inequality $\mu_2 < -\alpha_T/(v-u)$, we have $\varphi(u) > 0$ and

$$\frac{T_1 - T_0}{\varphi} q \frac{r_1\mu_1}{r_2\mu_2}(e^{\mu_2 x} - 1) \leqslant y_2(x, u) \leqslant 0. \tag{2.5}$$

Let $m(x, u) = \mu_1 \int_0^x e^{-\mu_1 s} n\,ds$. Then the first integral relation of (2.4) can be rewritten in the form

$$y_1 = \frac{T_1 - T_0}{\varphi}[(\varphi - q(1-m))e^{\mu_1 x} + q]. \tag{2.6}$$

It follows from this that $y_1(x, u)$ is bounded if and only if the following inequality is satisfied

$$|\varphi(u) - q(1 - m(x, u))| \leqslant \text{const}\, e^{-\mu_1 x}. \tag{2.7}$$

Let $m_\infty(u) = \lim_{x\to\infty} m(x, u) = \mu_1 \int_0^\infty e^{-\mu_1 x} n\,dx$. For $u < v/(1 + \sigma)$ this function is continuous and $0 \le m_\infty(u) \le 1$. In addition, $m_\infty(u) = 1$ if and only if $n(x, u) \equiv 1$. It can be shown that inequality (2.7) is equivalent to the following equality (from the point of view of bounding the solution):

$$\varphi(u) = q(1 - m_\infty(u)). \tag{2.8}$$

We do not discuss the existence of solutions to (2.8) for now, but show that this proof gives the solution of the problem stated in Sec. 1. Let $n \to n_e$ for $x \to +\infty$. Then $\mu_1 \int_x^\infty e^{-\mu_1 s} n\,ds \to n_e$, and from (2.8) it follows that we now show that

$$y_1 \to [(T_1 - T_0)/\varphi]q(1 - n_e). \tag{2.9}$$

We now show that if u satisfies (2.8), the temperature Θ is a monotonically increasing function of x. From (2.1) and (2.2), it follows that $d\Theta/dx = \mu_1 r_1 y_1 + \mu_2 r_2 y_2$. Substituting (2.4) into this relation and using (2.8) we obtain

$$\frac{d\Theta}{dx} = \frac{T_1 - T_0}{\varphi}\frac{\mu_1 r_1}{v-u} q\left[\int_x^\infty e^{\mu_1(x-s)} w(n, T)\,ds + \int_0^x e^{\mu_2(x-s)} w(n, T)\,ds\right].$$

Conditions (1.3) give the required proof. Because y_1 and y_2 are bounded, it follows that Θ and T are also bounded. From the fact that Θ is monotonic, it follows that $\Theta \to \Theta_e$. Because y_1 has a finite limit (2.9), then y_2 also has a finite limit, and hence $T \to T_e$. It is known (cf. [4]) that Θ_e, T_e, and n_e are stationary points. From the monotonicity of Θ we have $\Theta_e > \Theta(0, u) > T_1$, i.e., the stationary point is given by (1.6).

It remains to be shown whether (2.8) is soluble for $u < v/(1 + \sigma)$. It was shown in [5] that $\varphi(v/(1 + \sigma)) = 0$ and $\varphi(u) \to +\infty$ for $u \to -\infty$. Hence for $u_0 = v(1 + \sigma) - \delta$ we have

$\varphi(u_0) = O(\delta)$. The function $m_\infty(u)$ is representable in the form

$$m_\infty(u) = \mu_1 \int_0^\infty e^{-\mu_1 x - \frac{1}{v-u}\int_0^x \frac{w(n,T)}{n} ds} dx.$$

Using the fact that $\mu_1(u_0) = O(\delta)$ and $T(1, u_0) = T_1 + O(\delta)$, we obtain

$$m_\infty(u_0) \leqslant O(\delta)\left[1 + \int_1^\infty e^{-O(\delta)\left(1+k_0 w_T'(1,T_1)\right)x} dx\right].$$

We require that the following inequality be satisfied

$$w_T'(1, T_1 + 0) \geqslant k_1 > 0. \tag{2.10}$$

where k_0 and k_1 are positive constants. From this inequality it follows that, independently of δ, there exists a positive constant k_2 such that $m_\infty(u_0) \leq 1 - k_2$. It follows at once from this that (2.8) is soluble. The restriction (2.10) on the function w_T' is not overly severe. In our treatment, there is a certain arbitrariness in the cut-off point T_1 (in fact, we consider an Arrhenius heat source). Setting T_1 equal to its maximum possible value, the inequality (2.10) can be satisfied. Restrictions on T_1 are considered in [6].

Thus the existence of the solution to (1.1), (1.2) has been proven. The existence conditions are requirements on the reaction rate function (1.3), (2.10).

3. Wave Velocity at Large Activation Energies. A large activation energy for a chemical reaction means that $E/RT_1 \gg 1$. This in turn leads to the approximate relation $m_\infty(u) \approx 0$. Then the equation for the wave velocity takes the form

$$\varphi(u) = q. \tag{3.1}$$

It was shown in [5] that when $\sigma > 1$, the function $\varphi(u)$ monotonically decreases from ∞ to zero as u varies from $-\infty$ to $v/(1 + \sigma)$. It follows from this that the solution of (3.1) is unique. In [1] the same equation was obtained as a condition that the temperature of the solid phase Θ be continuous across the boundary between the heating and cooling regions. We note that the solution of (3.1) is a lower bound of the solution of (2.8). In the use of (3.1), it is essential that T_1 be the maximum possible value. In [1] this corresponds to the requirement that the chemical reaction range be narrow in comparison to the range of the temperature Θ. Then the transition between the heating and cooling regions in the solid phase can be considered as continuous. This means that if we take a cut-off point $T_1' < T_1$ and calculate the corresponding function $m_\infty'(u)$, one cannot put $m_\infty'(u) \approx 0$. Indeed, let $T(x_0, u) = T_1'$, where $x_0 < 0$. Then $q' = \frac{Q}{c_T}\Big|(T_1' - T_0) = e^{-\mu_1 x_0} q$. Further, $m_\infty'(u) = \mu_1 \int_{x_0}^\infty e^{-\mu_1(x-x_0)} n ds.$

Using the fact that $n(x, u) \equiv 1$ for $x_0 \leq x \leq 0$, we obtain $m_\infty'(u) = 1 - e^{\mu_1 x_0}(1 - m_\infty(u))$. Then we have

$$\varphi(u) = q(1 - m_\infty(u)) = q'\left(1 - m_\infty'(u)\right).$$

but if $m_\infty(u) \approx 0$ then $m_\infty'(u) \approx 1 - e^{\mu_1 x_0}$ and one cannot put $m_\infty(u) \approx 0$. Calculations using (3.1) done in [1] have shown its versatility over a wide range of parameters when compared to the experimental data.

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